Analytical Model of Inlet Growth and Equilibrium Cross-Sectional Area

by Richard Styles

PURPOSE: This Coastal and Hydraulics Engineering Technical Note (CHETN) focuses on inlet stability theory in the context of quantifying the conditions responsible for inlet growth, maintenance, or collapse. The theoretical underpinnings are drawn from the classic Escoffier (1940) inlet stability analysis to produce a new quadratic formula derived from simplified momentum and conservation equations describing maximum inlet flow velocity. While the models give similar results, they are formulated using different simplifying assumptions and thus encompass a broader range of theoretical alternatives to investigate inlet stability. The quadratic formula provides quantitative estimates of equilibrium cross-sectional area and depth and can be used to investigate the low-order dynamics of barrier island inlets, including breaches and other constricted flow regions that are driven by tidal forcing.

INTRODUCTION: Tidal inlet stability has long been of interest to coastal engineers and planners. Inlets connect the land to the sea and are primary conduits for seafaring commerce and provide significant environmental benefits in estuarine habitat, tidal exchange, and circulation. The U.S. Army Corps of Engineers is responsible for maintaining navigable waterways, and a lack of understanding of inlet stability can reduce navigability and increase the potential for storm damage. Storm-induced breaches that form new inlets can potentially pose hazards to coastal communities as well as adversely impact the delicate estuarine environment. Methodologies to predict the growth, maximum size, and flow rate of inlets can help determine the level of resources required to seal or stabilize breaches. This CHETN presents an analytical stability theory that serves as a low-order screening tool to diagnose the behavior and evolution of barrier island tidal inlets. Sensitivity to model input parameters is highlighted in the context of applications to real-world inlets as well as to specify user guidance for determining model limits and applicable range. A natural breach formed in the aftermath of Hurricane Sandy (2012) on Fire Island, NY, is used as a test case to illustrate model performance in a real-world setting.

BACKGROUND: Long-term inlet stability in bar-built systems is determined by the tidal and wave forces that rework sediment. Escoffier (1940) theorized that stable inlets form when the maximum flow equals an equilibrium value based on sediment transport processes. If maximum flow in the inlet (U) exceeds the minimum for the initiation of sediment motion, then the bed erodes and the cross-sectional area (A) increases. If the maximum flow is less than the equilibrium value, deposition occurs causing the inlet to close. Equilibrium is maintained when the forcing is energetic enough to mobilize recently deposited material but is unable to scour the underlying substrate. Escoffier (1940, 1977) constructed a theoretical framework to quantify the equilibrium approach using simplified momentum and continuity arguments for single inlet systems. Assuming the tidal wavelength is much greater than the length of the inlet or bay, the inlet velocity is approximately spatially uniform. The linearized one-dimensional (1D) momentum equation can be written as

$$\frac{\partial u}{\partial t} = -\frac{g}{L}(\eta - \eta_{oce}) - \frac{c_d u^2}{h} \qquad (1)$$

where u is the cross-sectionally averaged velocity in the inlet, t is the time, g is the acceleration due to gravity, L is the length of the inlet, η is the water surface elevation in the bay, η_{oce} is the water surface elevation in the ocean, h is mean water depth, and c_d is the drag coefficient. In earlier treatments, Escoffier used the hydraulic radius instead of water depth in the friction term (Walton and Escoffier 1981). However, the ratio of channel depth to channel width is typically very small ($1/10 - 1/100$) in natural channels (Yalin 1992). Sidewall effects are negligible compared to bed friction, and water depth is the appropriate scale in the friction term. The continuity equation for a simple bay system is written as

$$\frac{d}{dt}(A_b \eta) = uA \qquad (2)$$

where A_b is the bay area and A is the inlet cross-sectional area. Equation 2 describes the change in bay volume due to the discharge through the inlet throat. In general, A_b and A are functions of water depth, but a simple analytical solution emerges if the explicit dependence is overlooked. The set of equations can be solved to determine the maximum velocity through the inlet as a function of inlet cross-sectional area. Escoffier (1977) closed the solution by linearizing the friction term and neglecting time dependence, which led to expressions for the bay (a_b) and ocean (a_0) tidal amplitude ratio. The details can be found in his original work and are not presented here except to note that Escoffier introduced two empirical coefficients to close the momentum balance: (1) a Manning's n coefficient to represent friction and (2) a head-loss coefficient (m) to represent energy loss through the inlet throat.

Assuming a sinusoidal tidal current in the inlet throat, Equation 2 can be solved to give the maximum bay water surface elevation,

$$\eta_{bm} = a_b = \frac{A}{2A_b} \int_0^{T/2} U \sin(\omega t)\, dt = \frac{AU}{\omega A_b} \qquad (3)$$

Rearranging, the maximum inlet velocity is written as

$$U = \frac{\omega A_b a_b}{A} \qquad (4)$$

where $\omega = 2\pi / T$ is the radian frequency and T is tidal period.

Escoffier (1940) noted that to obtain a solution that remains finite as $A \to 0$, the water depth must be expressed in terms of A. A dimensionally correct approach is to write $h = \sqrt{\alpha A}$, where the nondimensional proportionality coefficient α is constant. For a rectangular channel, α is the ratio of inlet depth to width and is typically on the order of 0.1 to 0.01.

As an alternative to the Escoffier (1977) solution, the maximum current can be obtained directly from the momentum equation without invoking assumptions regarding inlet head loss or

linearization of the friction term. Substituting Equation 3 into Equation 1, neglecting time dependence and taking the maximum current gives the following quadratic equation:

$$U^2 + \frac{gAh}{c_d \omega L A_b}U - \frac{gha_0}{c_d L} = 0 \tag{5}$$

with the solution

$$U = \sqrt{B^2 A^2 + 2Ba_0 A_b \omega} - BA \tag{6}$$

where

$$B = \frac{gh}{c_d \omega L A_b} \tag{7}$$

The drag coefficient is a function of the total bottom roughness, including form drag and friction due to bedforms. Typical values for sandy coastal areas range between 5.0×10^{-4} for very smooth conditions to 5.0×10^{-2} for rough beds. Values on the order of $3.0 - 15.0 \times 10^{-3}$ are widely reported in the literature (Nielsen 1992).

A simple model to predict equilibrium cross-sectional area using a combination of the Escoffier and O'Brien equations was developed by Seabergh and Kraus (1997) and Seabergh et al. (2001). They expressed tidal prism as the product of bay area and tidal range in the bay, which can be determined by manipulating Equation 4 to produce the following equation for tidal prism:

$$P = \frac{2UA_e}{\omega} \tag{8}$$

Equating this expression to the original tidal prism relationship developed by O'Brien (1939),

$$P = aA_e^c \tag{9}$$

allows the equilibrium cross-sectional area (A_e) to be written in terms of the velocity, which is derived from Equation (6) or the Escoffier solution. Since O'Brien's original work, other investigators have calibrated the empirically derived parameters, *a* and *c*, using newer data sets obtained for U.S. inlets (Kraus 1998; Stive and Rakhorst 2008). One issue with this formulation is that both the Escoffier and quadratic method use Equation 4 to derive *V* and then use Equation 9 to compute A_e. If the tidal prism is given, then it is not necessary to invoke the Escoffier or quadratic approach as the equilibrium area can be determined through Equation 9. As an alternative, cross-sectional equilibrium is expressed in terms of sediment transport criteria.

Shields Criteria. The conceptual model of inlet stability is based on the idea that cross-sectional area growth/contraction results from scour/deposition in the inlet throat. The equilibrium cross section is maintained when the maximum shear stress equals that required to initiate sediment motion.

Initiation of sediment motion is determined using the Shields criteria, i.e.,

$$\psi = \frac{|\tau_b'|}{\rho(s-1)gd} \tag{10}$$

where $|\tau_b'|$ is the magnitude of the maximum skin friction shear stress, ρ is the fluid density, s is the specific gravity ($= \rho_s/\rho$, ρ_s is sediment density), and d is the representative grain diameter. Equilibrium is achieved when the Shields parameter (ψ) equals the critical value for initiation of sediment motion (ψ_{cr}). Further increases will mobilize sediment, potentially deepening the inlet. In practice, it is easier to measure the current so the magnitude of the skin friction shear stress is expressed in terms of a quadratic drag law,

$$|\tau_b'| = \rho c_s U^2 \tag{11}$$

where c_s is the drag coefficient based on skin friction and is a function of grain size. Substituting Equation 11 into Equation 10 gives the maximum Shields parameter,

$$\psi = \frac{c_s U^2}{(s-1)gd} \tag{12}$$

The skin friction drag coefficient is determined assuming a logarithmic boundary layer profile,

$$c_s = \frac{\kappa}{\ln\left(\dfrac{z_1}{z_0}\right)} \tag{13}$$

where κ is von-Karman's constant (0.4), z_0 is the hydraulic roughness, and z_1 is a suitably chosen height above the bed that represents the near-bed current. In the present application, z_1 is determined by linear regression using measured and predicted equilibrium cross-sectional area. Adopting the Nikuradse equivalent roughness (k_b), $z_0 = k_b/30 = d/30$. Note that for mixed beds, k_b is expressed in terms of some representation of the grain size distribution to ensure the correct skin friction roughness.

RESULTS

Maximum Current and Inlet Cross-Sectional Area. The Escoffier solution and the quadratic Equation 6 relating the maximum tidal current to cross-sectional area are qualitatively similar (Figure 1). For small A, the inlet velocity increases as flow drag decreases, and the cross section expands to accommodate greater exchange. For large A, the inlet throat is wide, and the flow is no longer constricted leading to a reduced sea surface slope across the inlet and lower current speeds. For typical values of drag coefficient and Manning's n, the Escoffier solution predicts a higher maximum current, but the qualitative features are the same (i.e., initial rapid growth to a peak current followed by asymptotic decay [$U \sim 1/A$]). In the Escoffier solution, the head loss coefficient, m, is a measure of energy loss through the constricted inlet throat and

therefore sets the upper bound on total energy loss. In other words, increases in n will reduce the velocity envelope only to a point dictated by m. Further increases in n will not affect the solution, and only increases in m will reduce the current (Figure 1).
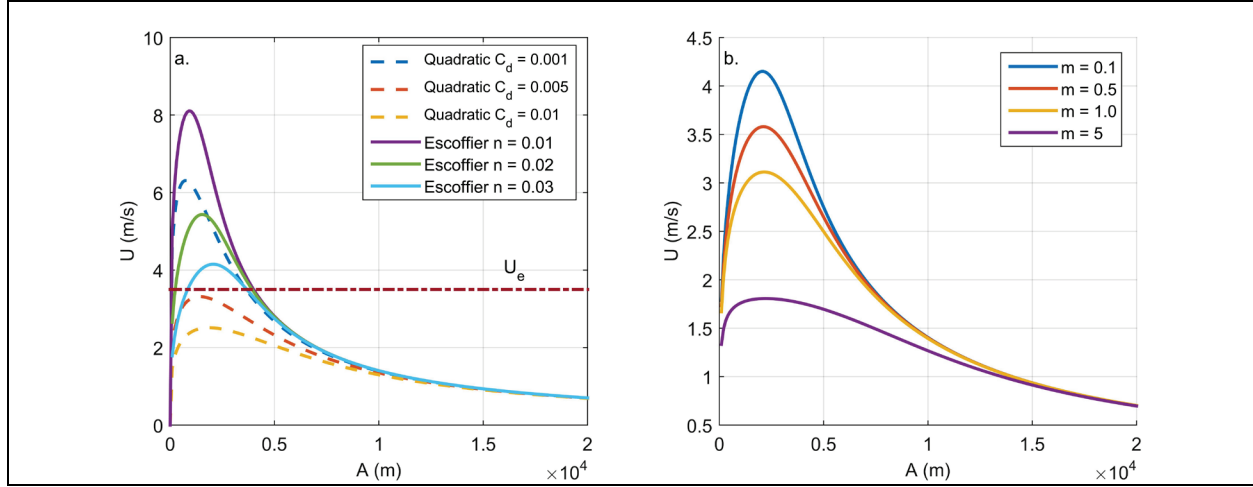

Figure 1. Theoretical example illustrating the quadratic and Escoffier solutions. The horizontal dash-dot line denotes the equilibrium velocity for a hypothetical inlet. Input parameters: L = 1000 m; a_0 = 1 m; A_b = 1.0 × 10⁸ m². (a.) Model sensitivity to drag coefficient and Manning's n. (b.) Escoffier model sensitivity to head loss coefficient, m.

The cross-sectional area that produces the maximum current denotes the transition between stable and unstable regimes. Taking the derivative of Equation 6 with respect to A, setting that expression equal to zero, and solving for A gives the cross-sectional area associated with the maximum current,

$$A_{u\_max} = \left(\frac{c_d L a_0 A_b^2 \omega^2}{6\alpha g} \right)^{2/5} \qquad (14)$$

A_{u_max} serves as a useful diagnostic for identifying the transition between stable and unstable conditions. If $A < A_{u\_max}$, then the inlet is unstable and small positive perturbations in the maximum velocity encourage inlet growth while small negative perturbations encourage inlet closure. A_{u_max} increases as a function of bay size and inlet length so that larger systems require larger cross-sectional area to remain stable, which is in agreement with the established O'Brien inlet stability theory (e.g., Hughes 2002; Kraus 1998; O'Brien 1939).

Escoffier theorized a constant equilibrium velocity (U_e), which would maintain a stable inlet. A hypothetical equilibrium velocity in which $U_{max} > U_e$ illustrates the role of the current in modulating the stability of the system (Figure 1). The first intersection point denotes an unstable inlet as small perturbations lead either to higher speeds and further expansion (erosion) of the throat, or lower speeds and closure (deposition). The second intersection point denotes a stable inlet as small perturbations tend to either decrease the current, causing deposition and a return to equilibrium as the larger cross section contracts, or increase the current, causing erosion and a return to equilibrium as the smaller cross section dilates. If $U_e > U_{max}$, then there is no choice of A that will produce strong enough currents to maintain a stable system, and the inlet will

eventually close. In Escoffier's original study, U_e was set equal to 1 m/sec (Seabergh and Kraus 1997), which through Equation (8) recovers the tidal prism relationship with A linearly proportional to P. In this study, U_e is also determined based on sediment transport criteria.

Equilibrium Cross-Section Predictions. The Coastal Inlets Research Program (CIRP) has developed a national inlet database consisting of information on tidal range, cross-sectional area, tidal prism, channel depth, and other physical variables. The record presently contains input variables needed to run the model for 54 inlets. The results indicate that individual inlet predictions are highly variable, yet both models (Escoffier and Quadratic) show little bias and predict the average conditions well (Figure 2).
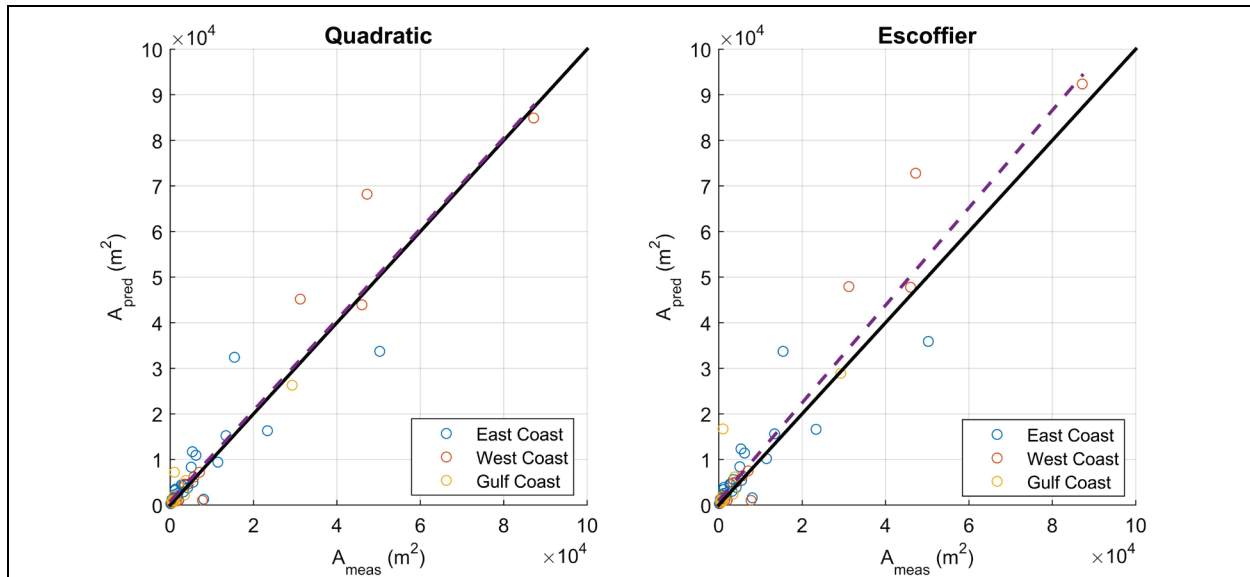

Figure 2.   Model predicted equilibrium inlet cross-sectional area comparison with U.S. inlets. Black line denotes 1:1 ratio. Dashed line denotes best fit (correlation coefficient, R = 0.95 Quadratic and 0.93 Escoffier).

The equilibrium cross-sectional area is determined as the intersection of Equation 6 and the current associated with the initiation of sediment motion as defined by the Shields criteria. The Shields parameter is based on U, which represents the depth-averaged flow as opposed to the current closer to the bed. Therefore, U overpredicts the current responsible for initiating sediment motion through Equation 13 unless a transfer function is used to extrapolate the current closer to the bed (Mofjeld 1988). Assuming a logarithmic velocity profile in which the current decreases towards the bed, it is possible to estimate the current that more accurately represents the near-bed flow responsible for sediment transport given bed roughness. At a height of 0.25 m above the bed, the logarithmic profile produces an average current that is 30% less than U, which is more representative of the near-bed flow (Nielsen 1992). Applying a lower current to compute the Shields parameter removes the bias error (the systematic overprediction of the cross section), but the source of random error depends on other factors. Simplifications in the model formulation and uncertainties in the input parameters (i.e., z_0) are likely sources of random error.

Model Parameter Uncertainty. The quadratic solution includes empirical drag coefficients for momentum and sediment, as well as the input variables for tidal amplitude, bay area, and inlet

length. In practice, some of these parameters are poorly constrained. For example, bay area can be difficult to quantify in multiple inlet systems, as the lateral boundary coincides with a tidal convergent node rather than a bounding shoreline. Tidal range is generally easier to estimate as accurate tidal information is abundant for many regions in the form of spring/neap time series or tidal constituent data bases. Inlet length can be measured from maps, but many inlets have complex shorelines owing to coastal sediment transport processes. For example, barrier inlets can be asymmetric between the up-drift and down-drift shoreline making inlet length difficult to measure. Likewise, jetties increase the effective length of inlets, so defining inlet length for one- and two-jettied systems requires an approach that characterizes the 1D channelized flow assumption. Flow drag and sediment grain size can also be difficult to prescribe as field data are often sparse or nonexistent for a particular inlet system and rarely characteristic of the median of sediment grain sizes transported. Bed roughness is proportional to grain size so that increases in grain size lead to increases in friction. As a result, the model behaves similarly to variations in these two parameters.

Parameter uncertainty is an important consideration during model setup, and the user should adopt a consistent approach that includes modulating the parameter space by considering a judiciously chosen range of inputs. In other words, the model should be run using a range of inputs that encompass the possible uncertainty in parameter values, and the resulting output extremes viewed as statistical error bounds.

Based on the comparison between modeled and predicted equilibrium cross-sectional area, the empirical parameters in the Escoffier and quadratic models are defined as $n = 0.02$, $m = 1$, $c_d = 0.003$, $z_1 = 0.25$ m. If data are available to calibrate the model, then these parameters should be adjusted accordingly.

Field Example. During Hurricane Sandy, a breach formed along the Long Island Coast near the eastward end of Fire Island, NY. The breach, referred to as Wilderness Breach, was allowed to evolve naturally, permitting the United States Geological Survey (USGS) and the USACE to conduct a series of field programs to monitor the expansion (*http://ny.water.usgs.gov/projects/ hurr_sandy_2012*). Starting with the initial cut in November 2012 and ending in October 2014, 14 bathymetric surveys and 4 current surveys were completed. The surveys provided a time history of cross-sectional area growth and flow speed as the unstable breach widened and deepened under natural forcing.

Input variables to drive the inlet stability model were determined as follows: Bay area – Measuring bay area in a multi-inlet system was difficult because the bay was hydraulically connected to the sea via two outlets. A first-order approach was to assume that bay filling was more or less equally divided among adjacent inlets and that a corresponding tidal node existed at the midpoint between neighboring inlets. Exchange across the node was limited due to convergence of the two incoming tidal waves. As such, the equivalent bay area was calculated using the midpoint as a boundary as this approach produced a tidal prism consistent with the O'Brien theory. An equivalent bay area of 4.6×10^7 m^2 was computed using the polygon feature in Google Earth (estimated error is on the order of 1×10^4 m^2 [Potere 2008]). Inlet length – inlet length was determined by measuring the distance from the entrance to the back bay shortly after the breach formed ($L = 300$ m). Tidal amplitude – tidal amplitude was determined from time series collected during the deployments, which indicated a maximum amplitude of 0.35 m. Grain size – a single grain size class of 400 μm (0.4 mm) was determined from beach samples collected on Fire Island (Dias and Sperazza 2014).

The maximum predicted current (quadratic formula) was 2.4 m/s, which corresponds to A_{u_max} = 308 m^2 (Figure 3). The measured cross-sectional area within the first 4 months after the breach formed was < 200 m^2, characteristic of an unstable inlet that continued to grow so long as the maximum current increased. The breach then increased over the next year to 620 m^2, indicating a transition from unstable to stable conditions. Currents likewise increased to a maximum of 2.4 m/s as the cross-sectional area increased to A = 253 m^2 and then decreased to 2.1 m/s when A exceeds A_{u_max}. The most recent survey in October 2014 measured a cross-sectional area of 640 m^2. The curve indicated that the inlet was in the stable regime, but it had not reached its theoretical maximum cross section (3450 m^2) so that further modifications were possible. For comparison, Fire Island Inlet located 34 km west of Wilderness Breach had a cross-sectional area of 3200 m^2. The Escoffier solution likewise predicted similar behavior as the quadratic formula, but maximum currents were smaller.
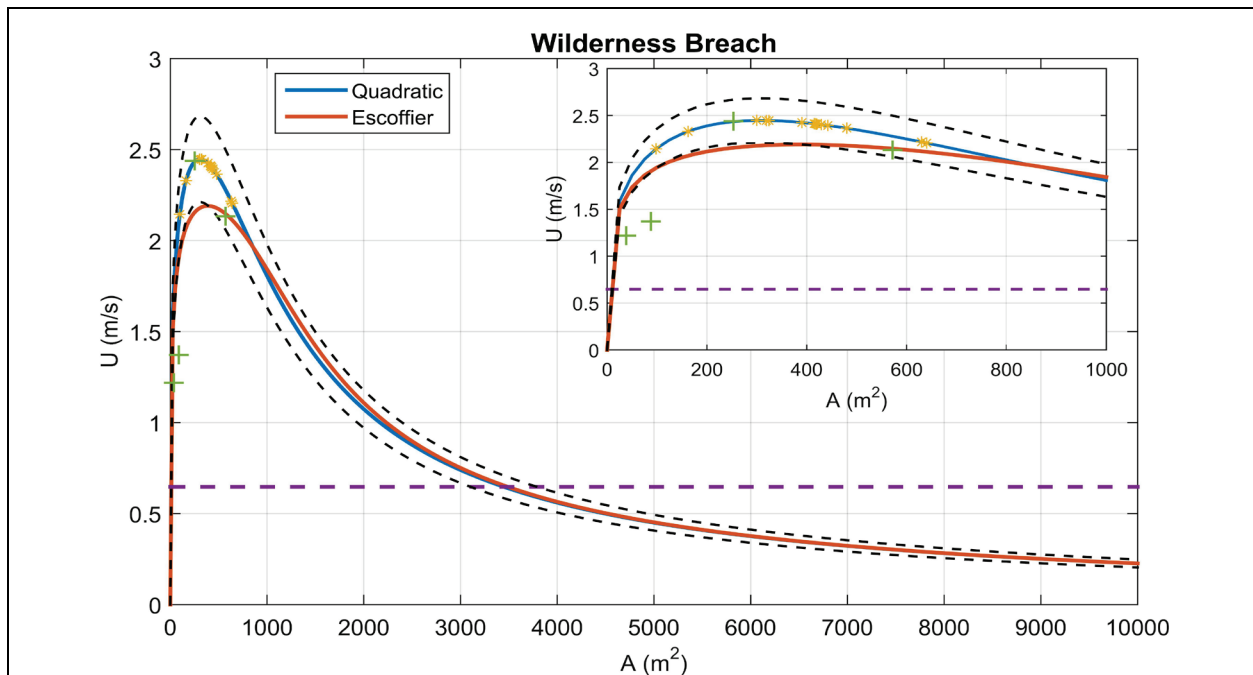

Figure 3.   Model predictions from Wilderness Breach, Fire Island, NY. Horizontal dashed line denotes current speed required to initiate sediment motion. Asterisks identify the theoretical current speed when the inlet cross-sectional area was measured, and (+) denotes measured currents. Velocity envelope error bounds (--) are based on the random error (R^2 = 0.9) Inset: Higher-resolution plot depicting conditions during surveys, for clarity.

DISCUSSION

Model Limitations. The theoretical model is based on simplifying assumptions that do not include higher-order processes active in real inlet systems (i.e., nonlinearity, co-oscillating tides, waves, along-shore transport). Therefore, the model is designed to illustrate the lowest-order processes active at inlets and to provide guidance for managing projects related to inlet stability including breaching studies. Fluid dynamical and geomorphological conditions at real inlets are very complex, involving nonlinear feedbacks and poorly constrained boundary conditions. In many cases, databases for inlet cross-sectional area, grain size distribution, inlet length, and bay area are sparse or incomplete, making it necessary to estimate these parameters in practice. Inlet

dynamics remains an active area of research, and new information is needed to improve upon the physics while maintaining a simple approach that can be rapidly and efficiently implemented.

The model does not explicitly include long-shore sediment transport, which is primarily driven by waves. Rather, the theory is founded on the principle that sediment transport is driven by tides and does not incorporate other forcing and associated sediment pathways. The ratio of wave to tidal energy is an important factor as wave-driven sediment transport can introduce large quantities of sediment into the system that can constrict the throat resulting in instabilities that may lead to inlet closure. However, in tide-dominated regions, the model is likely to lead to predictions that are more accurate.

The analysis is applicable to bar-built barrier islands with a significant sediment supply. Other inlets that are sediment starved (i.e., formed on rocky coasts, hard bottoms) or are dominated by fluvial inputs may not achieve equilibrium through tidal forcing alone.

CONCLUSIONS: A simple inlet stability theory has been presented based on simplifying assumptions for momentum, mass, and sediment transport. The work is an extension of previous equilibrium cross-sectional studies (Seabergh and Kraus 1997) but incorporates both empirical information (O'Brien relationship) and sediment transport processes in formulating the theory. The model calculates the cross-sectional area that defines the transition from unstable to stable conditions (maximum flow speed) and the maximum cross-sectional area based on sediment transport conditions. Beyond the stable inlet transition, the cross-sectional area may continue to grow but at the expense of reduced kinetic energy and less sediment transport potential. Eventually, equilibrium is reached in which currents are too weak to initiate sediment motion and further inlet growth is not possible. The model shows good agreement between modeled and measured inlet cross-sectional area, but the results do show significant scatter. This is attributed in part to the simplified approach and the fact that some inlets are jettied or dredged and reinforces the idea that the present models should primarily be used to describe the first-order behavior of inlets as appropriate for planning or screening studies. Test case results reveal that cross-sectional area growth and flow velocity are consistent with measurements obtained at a natural breach.

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Questions regarding this CHETN may be addressed to Richard Styles (*richard.styles@ usace.army.mil*). Additional information pertaining to the CIRP can be obtained from the USACE CIRP Program Manager, Julie Dean Rosati (*julie.d.rosati@usace.army.mil*).

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